C^k WEAKLY HOLOMORPHIC FUNCTIONS ON AN ANALYTIC CURVE

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Let V be a complex analytic set and let $x \in V$. Let $\mathcal{O}(V)$, $\tilde{\mathcal{O}}(V)$, $C^k(V)$, and $C^\infty(V)$ denote respectively the rings of germs at x of holomorphic, weakly holomorphic, k-times continuously differentiable, and infinitely differentiable functions on V. In [7] Malgrange proved that $C^\infty(V) \cap \tilde{\mathcal{O}}(V) = \mathcal{O}(V)$. Spallek [9] improved this result by showing that for sufficiently large k, $C^k(V) \cap \tilde{\mathcal{O}}(V) = \mathcal{O}(V)$. We will let k denote the smallest integer for which $C^k(V) \cap \tilde{\mathcal{O}}(V) = \mathcal{O}(V)$. It remains an open question to determine k. In the special case when x is an isolated singularity, Stutz [10] has given an upper bound for k in terms of some properties of the singularity of V by looking at the degree of the differential operators on the normalization of V which do not push forward to differential operators on V. In a recent thesis, Martha Jaffe [6] has refined the argument of Bloom [2] and Stutz [10] to its ultimate by determining all the differential operators involved on the curves $z^p = w^q$ in $\mathbf{C}^2$ with p and q relatively prime. These arguments prove that $k \leqq 3(p-1)(q-1)/2$. Bloom [3] has given an example where $k > 1$.

In this paper we study curves in $\mathbf{C}^2$ which have a normalization of the form $t \to (t^q, t^p u(t))$ where p and q are relatively prime integers, $p > q$, and $u(0) \neq 0$. For the case when $u \equiv 1$ our result is the following:

Theorem A. *$C^k(V) \cap \tilde{\mathcal{O}}(V) = \mathcal{O}(V)$ if and only if $k \geqq p(q-2)/q$.*

Let $D^k(V)$ denote the rings of germs at x of functions on V which are the restrictions to V of functions which are k-times continuously differentiable near V, and moreover are holomorphic in a neighborhood of the regular points of V. Clearly $D^k(V) \subset C^k(V) \cap \tilde{\mathcal{O}}(V)$ and it might be thought that the opposite inclusion is true as well. This is not true in general. In fact for the curves under consideration here, we have the following result:

Theorem B. *$D^k(V) = \mathcal{O}(V)$ if and only if $k \geqq q-1$.*

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In section 1 we find a basis for the finite $\mathbf{C}$-module $\tilde{\mathcal{O}}(V)/\mathcal{O}(V)$. In section 2 we prove Theorem B and in section 3 Theorem A. In each case the theorem is proved by determining the precise differentiability class of each basis function determined in section 1. In section 4 we prove a result which gives a differentiability condition which is sufficient to ensure that a weakly holomorphic function is holomorphic. This result is valid for arbitrary curves.

§1. The germ at the origin of any irreducible analytic curve V in $\mathbf{C}^2$ can be normalized by a map $\theta(t) = (t^q, t^p u(t))$, where $p > q$, q does not divide p, and $u(t)$ is a holomorphic function with $u(0) = 1$. If V is singular at the origin, that is if $q \geqq 2$, then p and q are invariant under biholomorphic change of coordinates. We will restrict our attention to the case where p and q are relatively prime, although this does not include all irreducible varieties (e.g., the curve normalized by $t \to (t^4, t^6 + t^7)$).

A monic holomorphic polynomial $P(z, w) = w^q + a_1(z)w^{q-1} + \cdots + a_q(z)$ such that V is the zero locus of P can be constructed as follows: for each $z \in \mathbf{C}$, $z \neq 0$, there are q points $(z, \alpha_j) \in V$, where $\alpha_j(z) = z^{p/q}\omega^{pj}u(z^{1/q}\omega^j)$, and ω is a primitive qth root of unity. Let

$$P(z, w) = \prod_{j=1}^{q} (w - \alpha_j(z)) = \sum_{i=0}^{q} a_i(z)w^{q-i}; \quad \text{then}$$

$a_i = \sigma_i(\alpha_1(z), \cdots, \alpha_q(z)) \in \mathcal{O}(\mathbf{C} - \{0\})$, where σ_i is the elementary symmetric polynomial of degree i. The coefficients a_i are bounded as $z \to 0$ and hence extend to holomorphic functions on $\mathbf{C}$ by the Riemann removable singularity theorem.

We need some preliminary results about weakly holomorphic functions. A holomorphic function $v(z, w) \in \mathcal{O}(V)$ is said to be a universal denominator if $v\tilde{\mathcal{O}} \subset \mathcal{O}$. It is well known that $\partial P/\partial w$ is a universal denominator [8]; however, this turns out to be insufficient for our needs. Let I be the ideal of $\mathcal{O}(V)$ of all functions vanishing on Sing (V), and J the ideal of all universal denominators for V. Then $J \subset I$ and the zero locus of J and I are the same. So by the Hilbert Nullstellensatz, there is a positive integer n such that $I^n \subset J$. Since the origin is an isolated singularity, $z^n \in J$. Also, the projection $\pi\colon \mathbf{C}^2 \to \mathbf{C}$ defined by $\pi(z, w) = z$ induces a homomorphism $\mathcal{O}(\mathbf{C}) \to \mathcal{O}(V)$ making $\mathcal{O}(V)$ into a finitely generated $\mathcal{O}(\mathbf{C})$ module with generators $1, w, \cdots, w^{q-1}$. Hence for any weakly holomorphic function $f(z, w)$, $z^n f(z, w) = \sum_{i=0}^{q-1} b_i(z)w^{q-i-1}$ in $\mathcal{O}(V)$.

Even better results can be obtained by arguing more directly. Let $N = [p(q-1)/q]$, where $[x]$ is the largest integer which is less than or equal to x. Notice that $p-2 \geqq N \geqq q-1$. Write $p = [p/q]q + r$, where r is a positive integer and $0 < r < q$.

Lemma 1. *z^N is a universal denominator for V.*

Proof. Let $f(z, w)$ be a weakly holomorphic function on V; then $g(t) = f(\theta(t))$ is a holomorphic function on $\mathbf{C}$. Extend $z^N f$ to $\mathbf{C}^2$ by

$$\begin{aligned} h(z, w) &= z^N \sum_{j=1}^{q} \prod_{k \neq j} \frac{w - \alpha_k(z)}{\alpha_j(z) - \alpha_k(z)} f(z, \alpha_j(z)) \\ &= \sum_{i=0}^{q-1} b_i(z) w^{q-i-1} . \end{aligned}$$

The coefficients are given by

$$b_i(z) = z^N \sum_{j=1}^{q} \frac{\sigma_i(\alpha_1(z), \cdots, \hat{\alpha}_j(z), \cdots, \alpha_q(z))}{\prod_{k \neq j}(\alpha_j(z) - \alpha_k(z))} f(z, \alpha_j(z))$$

where hatted terms are deleted. The functions $b_i(z)$ are well defined and holomorphic for $z \neq 0$. It remains only to show that $z b_i(z) \to 0$ as $z \to 0$. Now making the substitution $z = t^q$, we have:

$$\alpha_j(z) = t^p \gamma_j(t),\ \gamma_j(t) = \omega^{jp} u(t\omega^j),\ f(z, \alpha_j) = g(t\omega^j)$$

$$z b_i(z) = \frac{t^{q(N+1)+pi}}{t^{p(q-1)}} \sum_{j=1}^{q} \frac{\sigma_i(\gamma_1, \cdots, \hat{\gamma}_j, \cdots, \gamma_q)}{\prod_{k \neq j}(\gamma_j - \gamma_k)} g(t\omega^j) ;$$

p and q are relatively prime so $\gamma_j(0) - \gamma_k(0) = \omega^{jp} - \omega^{kp} \neq 0$. Hence it suffices to point out that $q(N+1) > p(q-1)$.

Theorem 1. *If $f \in \tilde{\mathcal{O}}(V)$ there are unique constants $b_{\mu\nu}$ such that $f(z, w) - \sum_{\mu=1}^{q-1} \sum_{\nu=0}^{N-1} b_{\mu\nu} w^\mu z^{\nu-N} \in \mathcal{O}(V)$. Furthermore $b_{\mu\nu} = 0$ if $p\mu + q\nu < qN$.*

Proof. By Lemma 1, $z^N f = \sum_{\mu=0}^{q-1} w^\mu b_{q-\mu-1}(z)$, so dividing by z^N and taking the power series expansion of each $b_{q-\mu-1}(z) = \sum b_{\nu\mu} z^\nu$ we have

$$f = \sum_{\mu=0}^{q-1} \sum_{\nu=0}^{N-1} b_{\nu\mu} w^\mu z^{\nu-N} + h(z, w), \quad h \in \mathcal{O}(V).$$

Composing with θ yields that $\sum_{\mu=0}^{q-1} \sum_{\nu=0}^{N-1} b_{\nu\mu} u(t)^\mu t^{p\mu+q\nu-qN}$ is bounded as $t \to 0$. Let σ be the smallest value of $p\mu + q\nu - qN$ occurring in the above sum for which $b_{\nu\mu} \neq 0$. If $\sigma < 0$, $t^\sigma \to \infty$ as $t \to 0$, so this term must be cancelled by another term in the sum with the same σ. But if $p\mu_1 + q\nu_1 - qN = p\mu_2 + q\nu_2 - qN$ with $1 \leqq \mu_j < q$, then necessarily

$\mu_1 = \mu_2$ and $\nu_1 = \nu_2$ because p and q are relatively prime. Hence $\sigma \geqq 0$ and any terms with $\mu = 0$ are impossible because $p0 + q(N-1) < qN\cdot$

To show that the constants are unique, it suffices to show that if $g(z,w) = \sum_{\mu=1}^{q-1}\sum_{\nu=0}^{N-1} b_{\mu\nu}w^{\mu}z^{\nu-N} \in \mathcal{O}(V)$, then necessarily all the coefficients are zero. Again we consider $g(\theta(t))$ and let σ be the smallest value of $p\mu + q(\nu - N)$ for which $b_{\mu\nu} \neq 0$. Since $g \in \mathcal{O}(V)$, the t^{σ} term in $g(\theta(t))$ must be cancelled by a holomorphic monomial $c_{ij}w^iz^j$. and consequently $\sigma = pi + qj$ with both i and j non-negative. As before, this is impossible since p and q are relatively prime.

Corollary. w^{q-1}/z is the most differentiable element of $\tilde{\mathcal{O}} - \mathcal{O}$.

Proof. Let $f \in \tilde{\mathcal{O}} - \mathcal{O}$ be represented as in Theorem 1, j be the lowest exponent in the sum $\sum b_{\nu\mu}w^{\mu}z^{\nu-N}$ such that $b_{j\mu} \neq 0$ for some μ, and i the lowest exponent so that $b_{ji} \neq 0$. Then letting $g = w^{q-i-1}z^{N-j-1}f$, we have $g = b_{ji}w^{q-1}/z +$ holomorphic terms so g is precisely as differentiable as w^{q-1}/z. But g is clearly at least as differentiable as f.

At this point with little extra effort we can give an algebraic interpretation of the number N.

Proposition 1. N is the smallest integer such that $I^N \subset J$.

Proof. Since $w^{q-1}/z^N \in \tilde{\mathcal{O}}(V)$ and $w^{q-1}/z \notin \mathcal{O}(V)$ it is clear that $z^{N-1} \notin J$. Thus no smaller integer will work.

To show the inclusion we must consider the polynomial $P(z,w)$ more carefully. We have $P(z,w) = \sum_{\nu=0}^{q} a_\nu(z)w^{q-\nu}$, and the coefficients are given by $a_\nu(z) = \sigma_\nu(\alpha_1(z),\cdots,\alpha_q(z))$ where $\alpha_j(z) = z^{p/q}\omega^{pj}u(z^{1/q}\omega^j)$. Thus $a_\nu(z) = O(|z|^{p\nu/q})$ as $z \to 0$.

As elements of $\mathcal{O}(V)$ we have therefore

$$w^q = -\sum_{\nu=1}^{q} a_\nu(z)w^{q-\nu}$$

$$w^{q+1} = -a_1w^q - \sum_{\nu=2}^{q} a_\nu w^{q-\nu+1}$$

$$= \sum_{\nu=1}^{q} (a_1a_\nu - a_{\nu+1})w^{q-\nu}$$

where we have set $a_\nu = 0$ if $\nu > q$. More generally we have $w^{q+l} = \sum_{\nu=1}^{q} c_\nu^l w^{q-\nu}$, where the coefficients satisfy the recurrence relations $c_\nu^0 = -a_\nu$, $c_\nu^{l+1} = c_{\nu+1}^l - a_\nu c_1^l$. By the above $a_\nu(z) = O(|z|^{p\nu/q})$ as $z \to 0$, and by induction it is easily seen that $c_\nu^l(z) = O(|z|^{p(l+\nu)/q})$ as $z \to 0$.

Now let $i + j \geqq N$. We wish to show that $w^i z^j$ is a universal denominator, and therefore that $w^i z^j w^\mu z^{\nu - N} \in \mathcal{O}(V)$ if $p\mu + q\nu \geqq qN$. This is immediate if $j + \nu \geqq N$, so suppose $j + \nu \leqq N-1$. Then $2N \leqq i + j + p\mu/q + \nu \leqq N - 1 + i + p\mu/q$, so $i + p\mu/q \geqq p(q-1)/q$, and finally $i + \mu > qi/p + \mu \geqq q-1$. Thus $w^{i+\mu} z^{j+\nu-N} = z^{j+\nu-N} \sum_{l=1}^{q} c_l^{i+\mu-q}(z) w^{q-l}$. This last expression is holomorphic if $j + \nu - N + p(i + \mu - q + l)/q > -1$ for $1 \leqq l \leqq q$. Since $j + \nu - N + p(i + \mu - q + 1)/q = (\nu + p\mu/q - N) + (j + pi/q) - p(q-1)/q \geqq j + i - p(q-1)/q \geqq N - p(q-1)/q > -1$, we are through.

Remark. We have actually shown that $w^i z^j$ is a universal denominator provided that $pi + qj \geqq qN$.

§2. Let $h_{\mu\nu} = w^\mu z^{\nu - N}$. Theorem B is a direct result of the following more precise result.

Theorem 2. *$h_{\mu\nu} \in D^k(V)$ if and only if $k \leqq \mu + q(\nu - N)/p$.*

To see that Theorem B follows from Theorem 2, we note first that $h_{q-1,N-1} \in D^{q-2}(V) \sim \mathcal{O}(V)$. To show that $D^{q-1}(V) = \mathcal{O}(V)$, suppose on the contrary that $f \in D^{q-1}(V) \sim \mathcal{O}(V)$. Then by the argument used in the corollary to Theorem 1, $h_{q-1,N-1} \in D^{q-1}$. This contradicts Theorem 2.

Proof. First we extend $h_{\mu\nu}$ to a function $H_{\mu\nu}$ which is holomorphic near $V \sim \{0\}$. Choose $\phi \in C^\infty(\boldsymbol{R})$ whose values are in the closed interval $[0, 1]$, such that:

$$\phi(t) = \begin{cases} 0 & \text{if } t \leqq 1/3 \text{ or } t \geqq 3 \\ 1 & \text{if } 1/2 \leqq t \leqq 2 \end{cases}.$$

Set $\Phi(z, w) = \phi(|z|^p / |w|^q)$. Then $\Phi \in C^\infty(\mathbf{C}^2 - \{0\})$. Let

$$N_\rho = \{(z, w) \colon |z|^p/\rho \leqq |w|^q \leqq \rho |z|^p\} \qquad \text{for } \rho > 1.$$

Then

$$\Phi(z, w) = \begin{cases} 1 & (z, w) \in N_2 - \{0\}. \\ 0 & (z, w) \notin N_3 \end{cases}.$$

There exists a ball B of some small radius about the origin such that $V \cap B \subset N_2$ because for $(z, w) = \theta(t)$,

$$|z|^p / |w|^q = 1/|u(t)|^q \to 1 \quad \text{as } t \to 0.$$

The function $H_{\mu\nu} = \Phi h_{\mu\nu} \in C^\infty(\mathbf{C}^2 - \{0\})$ since Φ equals zero where $h_{\mu\nu}$

blows up; also $H_{\mu\nu} = h_{\mu\nu}$ is holomorphic in $(N_2 \cap B) \sim \{0\}$ which is a neighborhood of $V \sim \{0\}$. We need only to check the differentiability of $H_{\mu\nu}$ near 0.

In a neighborhood of the origin, we have for $(z, w) \in N_3$: $|H_{\mu\nu}(z, w)| < 3^{\mu/q}|z|^{p\mu/q+\nu-N}$; the same estimate holds, of course, for $(z, w) \notin N_3$. Similarly

$$|D^\alpha H_{\mu\nu}(z, w)| \leqq C_\alpha |z|^{p\mu/q+(\nu-N)-(\alpha_1+\alpha_2)-p(\alpha_3+\alpha_4)/q}$$

where D^α is the differential operator $\left(\frac{\partial}{\partial z}\right)^{\alpha_1}\left(\frac{\partial}{\partial \bar{z}}\right)^{\alpha_2}\left(\frac{\partial}{\partial w}\right)^{\alpha_3}\left(\frac{\partial}{\partial \bar{w}}\right)^{\alpha_4}$ and α is the multi-index $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. It follows that $H_{\mu\nu} \in C^k(\mathbf{C}^2)$ for all $k = |\alpha| < \mu + q(\nu-N)/p$. We can take some extra continuous derivatives with respect to z, a total of $[p\mu/q] + \nu - N$ to be exact. However, it is easy to see this extension $H_{\mu\nu}$ does not have any extra continuous derivatives with respect to w because for $(z, w) \in N_2, D^\alpha H_{\mu\nu}$ has a lower bound similar to the above upper bound.

To show that $h_{\mu\nu}$ has no better extension we set $f(t) = h_{\mu\nu}(\theta(t)) = t^\sigma v(t)$, where $\sigma = p\mu + q(\nu-N)$ and $v(0) \neq 0$. Suppose $h_{\mu\nu} \in D^{[\sigma/p]+1}$ and set $f_\alpha(t) = \left(\frac{\partial}{\partial z}\right)^{\alpha_1}\left(\frac{\partial}{\partial w}\right)^{\alpha_2} h_{\mu\nu}(\theta(t))/\alpha!$ for multi-indices $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| \leqq K = [\sigma/p] + 1$. Then by Taylor's theorem

$$f(s) - \sum_{|\alpha| \leqq K} f_\alpha(t)(\theta(s) - \theta(t))^\alpha = o(|\theta(s) - \theta(t)|^K).$$

If we restrict to the lines $s = \omega^j t$ for $j = 1, 2, \cdots, K \leqq q-1$, where ω is a primitive qth root of unity, the only terms that remain in the above sum are those corresponding to multi-indices of the form $\alpha = (0, l)$ and we have

$$t^\sigma[\omega^{j\sigma}v(\omega^j t) - v(t)] - \sum_{l=1}^{K} f_{0,l}(t)t^{pl}[\omega^{jp}u(\omega^j t) - u(t)]^l = o(|t|^{Kp}).$$

Since $Kp > \sigma$ we have

$$\sum_{l=1}^{K-1} f_{0,l}(t)t^{pl-\sigma}[\omega^{jp}u(\omega^j t) - u(t)]^l \to (\omega^{j\sigma} - 1)v(0)$$

as $t \to 0$. Now the $(K-1) \times (K-1)$ matrix whose (j, l)th entry is $[\omega^{jp}u(\omega^j t) - u(t)]^l$ is invertible for small t (to see this we calculate the determinant directly by noting the similarity to the Vandermonde determinant; at $t = 0$ we get $\prod_{j=1}^{K-1}(\omega^{jp} - 1)\prod_{j<l}(\omega^{jp} - \omega^{lp}) \neq 0$). Consequently $\lim_{t\to 0} f_{0,l}(t)t^{pl-\sigma} = a_l$ exists for $1 \leqq l \leqq K-1$, and we have $\sum_{l=1}^{K-1} a_l(\omega^{jp}-1)^l$

$= (\omega^{j\sigma} - 1)v(0)$ for $1 \leqq j \leqq K$. It follows that the determinant of the matrix

$$\begin{pmatrix} (\omega^{\sigma}-1) & (\omega^{p}-1) & (\omega^{p}-1)^2 & \cdots & (\omega^{p}-1)^{K-1} \\ (\omega^{2\sigma}-1) & (\omega^{2p}-1) & (\omega^{2p}-1)^2 & \cdots & (\omega^{2p}-1)^{K-1} \\ \vdots & \vdots & \vdots & & \vdots \\ (\omega^{K\sigma}-1) & (\omega^{Kp}-1) & (\omega^{Kp}-1)^2 & \cdots & (\omega^{Kp}-1)^{K-1} \end{pmatrix}$$

is zero. However a computation, using a method suggested by R. Dunn, shows that the determinant is not zero.

The determinant is calculated by a series of row and column operations which will be presented in the form of an algorithm. Set $X_i = \omega^{pi} - 1$ for $i \geqq 0$, and $X_i = 0$ for $i \leqq 0$. Set $X^n_{i,j} = \sum_{|\alpha| = j-n-1} X_i^{\alpha_1} X_{i-1}^{\alpha_2} \cdot \cdots \cdot X_{i-n}^{\alpha_{n+1}}$; the sum is over all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_{n+1})$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} = j - n - 1$. Notice that in particular $X^n_{i,j} = 0$ if $j \leqq n$ and $X^n_{i,n+1} = 1$. In addition we have the fundamental relation

$$\begin{aligned} X^n_{i,j} - X^n_{i-1,j} &= (X_i - X_{i-n-1}) X^{n+1}_{i,j} \\ &= \omega^{p(i-n-1)}(\omega^{p(n+1)} - 1) X^{n+1}_{i,j} . \end{aligned}$$

The algorithm is as follows:

I. Subtract row $K - j$ from row $K - j + 1$ successively for $j = 1, 2 \cdots, K - 1$.

II. Factor $\omega^{\sigma} - 1$ from column 1.
Factor $\omega^{p} - 1$ from column j for $j = 2, \cdots, K$
Factor $\omega^{p(i-1)}$ from row i for $i = 1, 2, \cdots, K$.

At this point the matrix is $(Y^1_{i,j})$ where

$$Y^1_{i,j} = \begin{cases} \omega^{(i-1)(\sigma-p)} & j = 1 \\ X^1_{i,j} & j > 1 . \end{cases}$$

III. Set $n = 2$.

IV. Subtract row $K - j$ from row $K - j + 1$ successively for $j = 1, 2, \cdots, K - n + 1$.

Now the nth column of the matrix consists of all zeros except for 1 in the $(n-1)$st place. Thus the following step does not alter the determinant.

V. Set each element of the $(n-1)$st row equal to zero except for the nth entry (which is already $= 1$).

VI. Factor $\omega^{\sigma - p(n-1)} - 1$ from column 1.
Factor $\omega^{pn} - 1$ from column j for $j = n + 1, \cdots, K$.
Factor $\omega^{p(i-n)}$ from row i for $i = n, \cdots, K$.

The matrix is now $(Y^n_{i,j})$ where

$$Y^n_{i,j} = \begin{cases} \delta_{i,j-1} & \text{if } i < n \\ X^n_{i,j} & \text{if } i \geqq n \quad j > 1 \\ \omega^{(i-n)(\sigma-pn)} & \text{if } i \geqq n \quad j = 1 . \end{cases}$$

VII. If $n \leqq K-1$ set $n = n+1$ and go to IV.
If $n = K$ stop.

The final matrix is $(Y^K_{i,j})$ which has determinant $(-1)^K$. Taking into account all of the factors we see that the determinant of the original matrix is

$$(-1)^K \prod_{n=1}^{K} (\omega^{\sigma-p(n-1)} - 1)(\omega^{pn} - 1)^{K-n} \omega^{p(K-n)(K-n+1)/2} .$$

§3. For a restricted class of curves in $\mathbf{C}^2$ we can determine the precise differentiability class of each of the basis functions described in section 1.

Theorem 3. *Let V denote the curve in $\mathbf{C}^2$ defined by $z^p = w^q$ where p and q are relatively prime and $q < p$. Suppose $1 \leqq \mu \leqq q-1$ and $\sigma = p\mu - q\nu \geqq 0$. Then $h_{\mu\nu}(z,w) = w^\mu z^{-\nu} \in C^k(V)$ if and only if $k \leqq \max(\sigma/p, (\sigma-(p-q)(q-\mu))/q)$.*

Remark. Theorem A is an immediate consequence of Theorem 3 and the corollary to Theorem 1. We can extend part of Theorem A to a more general class of curves as follows.

Theorem 4. *Let $\theta(t) = (t^q, t^p u(t))$ where p and q are relatively prime, $q < p$, and u is a holomorphic function defined near 0 with $u(0) \neq 0$. Let $V = \{\theta(t) : t \in \mathbf{C}\}$. Then $C^k(V) \cap \widetilde{\mathcal{O}}(V) = \mathcal{O}(V)$ if $k > p(q-2)/q$.*

Proof of Theorem 3. Theorem 2 provides an extension of $h_{\mu\nu}$ to a neighborhood of V which belongs to C^k for $k < \sigma/\rho$. It remains to describe an extension which belongs to C^k for $k < (\sigma-(p-q)(q-\mu))/q$. Again we can write down such an extension. Choose $\phi \in C^\infty(\boldsymbol{R})$ which satisfies $\phi(t) = 0$ if $t \geqq 2$ and $\phi(t) = 1$ if $t \leqq 1$. Set $H_{\mu\nu}(z,w) = \phi(|w|/|z|)$ $|z|^{2p\mu/q} z^{-\nu} \bar{z}^{-p} \bar{w}^{q-\mu}$. It is easily seen that $H_{\mu\nu}$ is an extension of $h_{\mu\nu}$ and has the required differentiability.

To prove the converse, suppose $h_{\mu\nu}$ has an extension $F \in C^k$ where $k > \max(\sigma/p, (\sigma-(p-q(q-\mu))/q)$. Let P denote the Taylor polynomial of degree k for F at 0. Set $\theta(t) = (t^q, t^p)$ and define $f_\alpha(t) = D^\alpha(F-P)(\theta(t))$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $D^\alpha = \left(\frac{\partial}{\partial z}\right)^{\alpha_1} \left(\frac{\partial}{\partial \bar{z}}\right)^{\alpha_2} \left(\frac{\partial}{\partial w}\right)^{\alpha_3} \left(\frac{\partial}{\partial \bar{w}}\right)^{\alpha_4}$. Then

$$f_\alpha(t) = o(|t|^{q(k-|\alpha|)}),$$

$$f_\alpha(s) - \sum_{|\beta| \leqq k-|\alpha|} f_{\alpha+\beta}(t)(\theta(s)-\theta(t))^\beta/\beta! = o(|\theta(s)-\theta(t)|^{k-|\alpha|}).$$

$f(t) = f_0(t)$ is a real analytic function on $\mathbf{C}$ with a power series expansion $f(t) = \sum a_{ij} \; t^i \bar{t}^j$ when $a_{\sigma 0} \neq 0$ since σ is not of the form $pa + qb$ with a and b non-negative integers.

We wish to compare $f(\omega t)$ and $f(t)$ where ω is a primitive qth root of unity. This is impossible to do directly so instead we consider a weighted mean of f over all roots of unity. Set

$$g_\alpha(t) = 1/q \sum_{\nu=1}^{q} \omega^{\nu(p(\alpha_3-\alpha_4)-\sigma)} f_\alpha(\omega^\nu t).$$

Then we have

$$g_\alpha(\omega t) = \omega^{\sigma - p(\alpha_3-\alpha_4)} g_\alpha(t) \tag{3.1}$$

$$g_\alpha(t) = o(|t|^{q(k-|\alpha|)}) \tag{3.2}$$

$$g_\alpha(s) - \sum_{|\beta| \leqq k-|\alpha|} g_{\alpha+\beta}(t)(\theta(s)-\theta(t))^\beta/\beta! = o(|\theta(s)-\theta(t)|^{k-|\alpha|}). \tag{3.3}$$

In addition we have $g(t) = g_0(t) = \sum' a_{ij} t^i \bar{t}^j$, where the primed summation is over (i,j) for which $i - j \equiv \sigma (\operatorname{mod} q)$. Since $a_{\sigma 0} \neq 0$ we know that

$$g(t) \neq o(|t|^\sigma). \tag{3.4}$$

From (3.2) we see that

$$g_\alpha(t) = o(|t|^{\sigma - p|\alpha|}) \tag{3.5}$$

provided that $|\alpha| > (\sigma - qk)/(p-q)$. Since $k > \sigma/p$ it is clear that $k > (\sigma - qk)/(p-q)$ and we have (3.5) at least for $|\alpha| = k$. We will prove that (3.5) holds for all α by backwards induction on $|\alpha|$. Suppose (3.5) holds for all $|\alpha| \geqq l+1$, where $l \leqq (\sigma - qk)/(p-q)$. Then for $|\alpha| = l$ we have, by (3.1), (3.3) and the induction hypothesis, that

$$(\omega^{\sigma - p(\alpha_3-\alpha_4)} - 1) g_\alpha(t) = g_\alpha(\omega t) - g_\alpha(t) = o(|t|^{\sigma - p|\alpha|}).$$

The coefficient $\omega^{\sigma - p(\alpha_3-\alpha_4)} - 1$ is zero if and only if $\alpha_3 - \alpha_4 \equiv \mu (\operatorname{mod} q)$. This cannot happen if $|\alpha| = l < \min(\mu, q-\mu)$. Thus the induction will be complete provided that $(\sigma - qk)/(p-q) < \min(\mu, q-\mu)$. This is guaranteed by the hypothesis on k. Consequently we have (3.5) for $\alpha = 0$, which contradicts (3.4).

Proof of Theorem 4. The proof is similar to the proof of the converse part of Theorem 3. Suppose $k > p(q-2)/q$ and $C^k(V) \cap \tilde{\mathcal{O}}(V) \neq \mathcal{O}(V)$. Then by the corollary to Theorem 1, $h(z,w) = w^{q-1}z^{-1} \in C^k(V)$. Let $F \in C^k(\mathbf{C}^2)$ be an extension of h and let P be the Taylor polynomial of degree k for F at 0. Set $f_\alpha(t) = D_\alpha[F-P](\theta(t))$. Then $f(t) = f_0(t)$ is a real analytic function and has a power series expansion $f(t) = \sum a_{ij}t^i\bar{t}^j$. Let $\tau = p(q-1)-q$. Then either $a_{\tau 0} \neq 0$, or the t^τ term in $F(\theta(t))$ has to be cancelled by a term in $P(\theta(t))$. In this case $a_{\sigma 0} \neq 0$ for some $\sigma < \tau$ of the form $\sigma = p\mu + q\nu$, where $\mu > 0$ and $\nu \geqq 0$. In either case we have that there is a $\sigma \leqq p(q-1)-q$, such that $a_{\sigma 0} \neq 0$ with $\sigma \equiv p\mu (\operatorname{mod} q)$ and $1 \leqq \mu \leqq q-1$. On the other hand, if ω is a primitive qth root of unity we have

$$f(\omega t) - f(t) = \sum_{1 \leqq |\alpha| \leqq k} f_\alpha(t)[\theta(wt) - \theta(t)]^\alpha + o(|\theta(\omega t) - \theta(t)|^k).$$

Since $\theta(\omega t) - \theta(t) = (0, t^p(\omega^p u(\omega t) - u(t)))$ and $f_\alpha(t) = o(|t|^{q(k-|\alpha|)})$ we have $f(\omega t) - f(t) = o(|t|^{qk+p-q}) = o(|t|^{p(q-1)-q})$. This implies that $a_{\sigma 0} = 0$, a contradiction.

§4. For the general class of curves it is possible to give a differentiability condition which will ensure that a weakly holomorphic function is holomorphic. While this condition is not precise, it does have the advantage that it is expressed in terms of well-known invariants of the curve.

For a curve $V \subset \mathbf{C}^n$ with $\operatorname{sing}(V) = \{0\}$, let I denote the ideal of $\operatorname{sing}(V)$ and let J denote the ideal of universal denominators for V. The smallest integer N for which $I^N \subset J$ is called the *conductor number* of V (that N is finite is a consequence of the Hilbert Nullstellensatz).

Theorem 5. *Suppose V is an analytic curve. Let N be the conductor number of V. Then $C^N(V) \cap \tilde{\mathcal{O}}(V) = \mathcal{O}(V)$.*

Remark. It is an appealing conjecture that this result is true for all varieties (it is true for hypersurfaces [*1*]). If true this would imply Siu's result that the integer k for which $C^k(V) \cap \tilde{\mathcal{O}}(V) = \mathcal{O}(V)$ is bounded above on compact sets, because of the coherence of the ideal sheaves of I and J.

Remark. That the conductor number is not the best possible is clear from the previous sections. For the curve $z^p = w^q$ with p and q relatively prime, $N = [p(q-1)/q]$, while $C^k(V) \cap \tilde{\mathcal{O}}(V) = \mathcal{O}(V)$ if $k \geqq p(q-2)/q$.

We may assume that the curve $V \subset \mathbf{C}^n$ has a normalization of the form $\theta(t) = (t^{p_1}, t^{p_2}u_2(t), \cdots, t^{p_n}u_n(t))$ where $p_1 < p_j$, $2 \leqq j \leqq n$, and $u_j(0) = 1, 2 \leqq j \leqq n$.

The key to the proof is the following lemma which is taken from [9].

Lemma 2. *Suppose* $f \in C^k(V) \cap \tilde{\mathcal{O}}(V)$ *.Then there is a holomorphic polynomial* $H(z)$ *in* $\mathbf{C}^n$ *such that* $f(z) - H(z) = O(|z|^k)$ *for* $z \in V$.

Proof. Suppose $f = F/V$ where $F \in C^k(\mathbf{C}^n)$. Let P denote the Taylor polynomial of degree k for F at 0. Let $P(z, \bar{z}) = H(z) + A(z, \bar{z})$ where H is holomorphic and A contains no holomorphic terms. Then $\theta^*(f - P) = \theta^*(f - H) + \theta^*(A)$ is a real analytic function, $\theta^*(f-H)$ is holomorphic, and $\theta^*(A)$ has no holomorphic terms. Consequently, since $\theta^*(f - P)(t) = o(|t|^{kp_1})$, its holomorphic part $\theta^*(f - H)(t) = o(|t|^{kp_1})$. Thus $f(z) - H(z) = o(|z_1|^k) = o(|z|^k)$ on V.

Proof of Theorem 5. Let $f \in C^N(V) \cap \tilde{\mathcal{O}}(V)$. Then by lemma 2 there is a holomorphic polynomial H such that $\theta^*(f-H)(t) = o(|t|^{Np_1})$. Thus $\theta^*(f-H)(t) = t^{Np_1}g(t)$. If we write $g(t) = \theta^*G(t)$ where $G \in \tilde{\mathcal{O}}(V)$ we have $f - H = z_1^N G$ on V. Since z_1^N is a universal denominator, we have $f = H + z_1^N G \in \mathcal{O}(V)$.

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